A subspace fitting method based on finite elements for fast identification of damages in vibrating mechanical systems

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Abstract

In this paper, a method based on subspace fitting is proposed for identification of faults in mechanical systems. The method uses the modal information from an observability matrix, provided by a stochastic subspace identification. It is used for updating a Finite Element Model through the Variable Projection algorithm. Experimental example aims to demonstrate the ability and the efficiency of the method for diagnosis of structural faults in a mechanical systems.

1 Introduction

Detecting the damages in vibrating mechanical systems, at the earliest stage, is a crucial task in many industrial sectors. Emergence of cracks is one example of damages occurring in a mechanical system during its service life. Such phenomena can be understood as modifications of the modal parameters of the structure (say the eigenfrequencies, the wave shapes and the damping factors). Identifying these modal parameters and analyzing their variations in the time domain appears as an efficient means for detecting and localizing these damages within the structure.

In recent years, sensibility-based Finite Element Model (FEM) updating methods [1, 11, 12, 15] have been successfully used for damage assessment. In general, updating a FEM in structural dynamics consist in adjusting the variables of numerical model to those provided by experimental datas.

In this work, a technique which is based on the subspace fitting methods [6, 7, 13] is presented. The key idea behind subspace methods [4, 9, 14] is to consider a block Hankel matrix constructed from the input/output vibrational signals of a mechanical system; the procedure consists in projecting this Hankel matrix onto a subspace which is well suited for identifying the related modal parameters. The procedure also consists in formulating an observability matrix using the QR factorization and singular value decomposition (SVD) procedure. Subspace fitting methods differ from the conventional subspace methods in exploiting the whole observability matrix, rather than applying a simple shift invariance procedure. This yields the modal parameters of the mechanical system to be identified accurately in a least squares sense.

The prosed method aims at using any a priori information about the system dynamic which is delivered from a FEM can be incorporated into the problem. The theoretical observability matrix can be built and is accounted for to solve the problem. This error between the experimental and numerical solutions can be minimized through modal parameters using the Variable Projection (VP) algorithm [2, 5, 23, 24].

The paper is organized as follow. In section two, the basics of the proposed method are presented. The subspace fitting method is explained. In section three, an optimized algorithm is proposed which enables the FEM of the system to be updated using the VP algorithm. Considering the damages of the system is discussed in section four. The procedure allows the proposed method to be applied to a structural health monitoring. An experimental experiment is finally presented in section five, which show the capabilities of the method to detect, localize and diagnosis damages in an Euler Bernoulli beam.

2 Objective function

Subspace fitting [6] is a concept that exploits, the inherent shift structure of the observability matrix. The observability matrix is obtained through a subspace identification technique, but in a different way compared to the classical subspace methods. The objective is to find a FEM that best fit, in the least square sense, the experimental data that represents the vibrating behavior of a complex mechanical system. The subspace fitting is based on the following relation

$$\mathbf{\Gamma_{exp}} = \mathbf{\Gamma}(\theta)\mathbf{T}, \tag{1}$$

where $\mathbf{\Gamma_{exp}}$ is the discrete observability matrix issue from a subspace identification procedure . $\mathbf{\Gamma}$ is the theoretical observability matrix that depend on $\{\theta_j\}_j$, that is a set of parametrization parameters. Just as there are many realizations or coordinates systems that can be used to describe the state-space, there are many identifiable parametrizations θ_j that can be chosen [7], each yielding a different $\mathbf{T}$ that satisfies Eq.(1).

The proposed method aims at establishing a theoretical observability matrix, which is based from a finite element model of the considered mechanical system. Using these FEM the modal parameters are obtained by solving the following quadratic eigenproblem

$$(\mu_j^2\mathbf{M} + \mu_j\mathbf{C} + \mathbf{K} + i\mathbf{D})\mathbf{\Phi}_j = 0, \tag{2}$$

where $\mathbf{M}$, $\mathbf{K}$, $\mathbf{C}$ and $\mathbf{D} \in \mathbb{R}^{n\times n}$ are the mass, stiffness, viscous damping and structural damping matrices of the FEM respectively. The solution of the eigenproblem are $\{\mu_j, \mathbf{\Phi}_j\}$, the continuous complex modal parameters composed of complex eigenvalues $\in \mathbb{C}^{2n\times 2n}$ and eigenvectors $\in \mathbb{C}^{n\times 2n}$ respectively.
Sampling FEM at rate Δt yields the discrete stochastic modal state space form

$$\begin{aligned} \mathbf{q}_{k+1} &= \mathbf{\Lambda}\mathbf{q}_k + \mathbf{w}_k & (3)\\ \mathbf{y}_k &= \mathbf{\Phi}^{obs}\mathbf{q}_k + \mathbf{v}_k & (4) \end{aligned}$$

where $\mathbf{\Lambda} = e^{\boldsymbol{\mu}\Delta t}$ and $\mathbf{\Phi}^{obs} \in \mathbb{C}^{l_y\times 2n}$ are the complex mode shapes at the l_y output locations. $\mathbf{w}_k$ and $\mathbf{v}_k \in \mathbb{R}^{2n\times 1}$ are respectively process and measurement noise vectors. λ_j is connected to the jth modal frequency f_j and damping ratio ζ_j by

$$\{\lambda_j, \lambda_j^*\} = e^{\left(-2\pi f_j\zeta_j \pm i2\pi f_j\sqrt{1-\zeta_j^2}\right)\Delta t}. \tag{5}$$

From the state-space representation, the theoretical observability matrix is obtained by

$$\mathbf{\Gamma} = \begin{bmatrix} \mathbf{\Phi}^{obs} \\ \mathbf{\Phi}^{obs}e^{\boldsymbol{\mu}\Delta t} \\ \vdots \\ \mathbf{\Phi}^{obs}e^{(\alpha-1)\boldsymbol{\mu}\Delta t} \end{bmatrix}, \tag{6}$$

where α is an integral number which is chosen greater than twice the system order [4], also $\boldsymbol{\mu} = \mathrm{diag}_k\{\mu_k\}$. Due to the effects of noise, only an estimate of $\mathbf{\Gamma_{exp}}$, says $\hat{\mathbf{\Gamma}}$ can be obtained by subspace identification [14]. If one assumes that $\mathbf{\Gamma}$ depends on a set of scalar parameters $\theta = \{\theta_j\}$ yields the identification problem to be formulated as

$$\{\hat{\theta}, \hat{\mathbf{T}}\} = \min\|\mathbf{\Gamma_{exp}} - \mathbf{\Gamma}(\theta)\mathbf{T}\|_F^2, \tag{7}$$

where $\|.\|_F$ denotes the Froebenus norm.
This problem can be resolved with the VP algorithm. This algorithm uses the fact that $\mathbf{T}$ can be optimally expressed (for a θ fixed) as $\hat{\mathbf{T}} = \mathbf{\Gamma}(\hat{\theta})^+\mathbf{\Gamma_{exp}}$, where $\mathbf{\Gamma}(\theta)^+$ is the Moore-Penrose pseudo inverse $\mathbf{\Gamma}(\theta)^+ = \left(\mathbf{\Gamma}^H(\theta)\mathbf{\Gamma}(\theta)\right)^{-1}\mathbf{\Gamma}^H_{\mathbf{th}}(\hat{\theta})$. By using this solution for $\mathbf{T}$ the objective function to minimize can be expressed as

$$\hat{\theta} = \mathrm{argmin}||\mathbf{r}(\theta)||_2^2 \tag{8}$$

where

$$\mathbf{r}(\theta) = \mathrm{vec}\left\{(\mathbf{I} - \mathbf{\Gamma}(\theta)\mathbf{\Gamma}(\theta)^+)\mathbf{\Gamma_{exp}}\right\} \tag{9}$$

3 Optimization algorithm

The Gauss-Newton algorithm is used to solve the minimization problem for $\mathbf{r}(\theta)$. This algorithm is based on the following relation [25]

$$\theta_{f+1} = \theta_f - \beta_f \mathbf{H}^{-1}\mathbf{g}. \tag{10}$$

where β_f is a step size. $\mathbf{g}$ and $\mathbf{H}$ are, respectively, the Gradient and the Hessian matrices of $||\mathbf{r}(\theta)||_2^2$ defined as

$$g_i = 2\mathrm{Re}\left\{\mathbf{r}^H \frac{\partial \mathbf{r}}{\partial \theta_i}\right\}, \tag{11}$$

$$H_{ij} \approx 2\mathrm{Re}\left\{\frac{\partial \mathbf{r}^H}{\partial \theta_i}\frac{\partial \mathbf{r}}{\partial \theta_j}\right\}, \tag{12}$$

where [23]

$$\begin{aligned}
\frac{\partial \mathbf{r}(\theta)}{\partial \theta_j} &= \mathrm{vec}\left\{-\frac{\partial}{\partial \theta_j}\left(\boldsymbol{\Gamma}\boldsymbol{\Gamma}^+\right)\boldsymbol{\Gamma}_{\mathbf{exp}} = -\left(\frac{\partial \boldsymbol{\Gamma}}{\partial \theta_j}\boldsymbol{\Gamma}^+ + \boldsymbol{\Gamma}\frac{\partial \boldsymbol{\Gamma}^+}{\partial \theta_j}\right)\boldsymbol{\Gamma}_{\mathbf{exp}}\right\}, && (13)\\
&= \mathrm{vec}\left\{-\frac{\partial \boldsymbol{\Gamma}}{\partial \theta_j}\boldsymbol{\Gamma}^+\boldsymbol{\Gamma}_{\mathbf{exp}} - \boldsymbol{\Gamma}\left(-\boldsymbol{\Gamma}\boldsymbol{\Gamma}^+\frac{\partial \boldsymbol{\Gamma}}{\partial \theta_j}\boldsymbol{\Gamma}^+ + (\boldsymbol{\Gamma}^H\boldsymbol{\Gamma})^{-1}\frac{\partial \boldsymbol{\Gamma}^H}{\partial \theta_j}[\mathbf{I} - \boldsymbol{\Gamma}\boldsymbol{\Gamma}^+]\right)\boldsymbol{\Gamma}_{\mathbf{exp}}\right\}, && (14)\\
&= \mathrm{vec}\left\{\left\{-[\mathbf{I} - \boldsymbol{\Gamma}\boldsymbol{\Gamma}^+]\frac{\partial \boldsymbol{\Gamma}}{\partial \theta_j}\boldsymbol{\Gamma}^+ - \boldsymbol{\Gamma}(\boldsymbol{\Gamma}^H\boldsymbol{\Gamma})^{-1}\frac{\partial \boldsymbol{\Gamma}}{\partial \theta_j}[\mathbf{I} - \boldsymbol{\Gamma}\boldsymbol{\Gamma}^+]\right\}\boldsymbol{\Gamma}_{\mathbf{exp}}\right\}, && (15)\\
&= \mathrm{vec}\left\{-\left\{[\mathbf{I} - \boldsymbol{\Gamma}\boldsymbol{\Gamma}^+]\frac{\partial \boldsymbol{\Gamma}}{\partial \theta_j}\boldsymbol{\Gamma}^+ + \left([\mathbf{I} - \boldsymbol{\Gamma}\boldsymbol{\Gamma}^+]^H\frac{\partial \boldsymbol{\Gamma}}{\partial \theta_j}\boldsymbol{\Gamma}^+\right)^H\right\}\boldsymbol{\Gamma}_{\mathbf{exp}}\right\}. && (16)
\end{aligned}$$

This expression can be simplified to yield [2, 5, 24]

$$\frac{\partial \mathbf{r}(\theta)}{\partial \theta_j} \approx -\mathrm{vec}\left\{[\mathbf{I} - \boldsymbol{\Gamma}\boldsymbol{\Gamma}^+]\frac{\partial \boldsymbol{\Gamma}}{\partial \theta_j}\boldsymbol{\Gamma}^+\boldsymbol{\Gamma}_{\mathbf{exp}}\right\}, \tag{17}$$

since it is assumed that the second term can be neglected compared to the other term at the neighborhood of the optimum.
The Gauss-Newton algorithm is applied to identified the matrix $\boldsymbol{\Gamma}$ (see Eq.(6)). In the present framework, the matrix $\boldsymbol{\Gamma}$ is assumed to depend on a the complex eigenvalues and eigenvectors of the quadratic eigenproblem (Eq.(2)), namely $\{\mu_k\}$ and $\{\Phi_k^{obs}\}$. Also, these parameters are assumed to depend on a set of scalar parameters denoted as $\{\theta_j\}$. Such parameters may refer to the material characteristics of the structure. The strategy for identifying these parameters can be explained as follows

Step 1 Consider initial guests for the eigenvectors $\{\Phi_k^{obs}\}$ and minimize $||\mathbf{r}(\theta)||_2^2$ with constant eigenvectors. Extract parameters $\{\theta_j^1\}$;

Step f $\geq$ 2 Update the eigenvectors $\{\Phi_k^{obs}\}$ and the eigenvalues $\{\mu_k\}$ with respect to the parameters $\{\theta_j^{f-1}\}$ evaluated at step $f-1$. Minimize $||\mathbf{r}(\theta)||_2^2$, with respect to the eigenvalues only. Extract parameters $\{\theta_j^f\}$.

The fact that the eigenvectors are constant in the minimization problem, for each iteration, is considered here to circumvent the problem of mixing the derivatives of both scalar and vectorial parameters which could penalize the convergence rate of the algorithm. Considering Eq.(6), the derivative for $\boldsymbol{\Gamma}$ with respect to $\{\theta_j^f\}$ can be expressed as

$$\frac{\partial \boldsymbol{\Gamma}(\theta)}{\partial \theta_j} = \sum_k \frac{\partial \boldsymbol{\Gamma}(\theta)}{\partial \mu_k}\frac{\partial \mu_k}{\partial \theta_j}. \tag{18}$$

where

$$\frac{\partial \boldsymbol{\Gamma}}{\partial \mu_k} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\Phi}^{obs} \Delta t \, \mathrm{diag}_j \{\delta_{jk}\} e^{\boldsymbol{\mu} \Delta t} \\ \boldsymbol{\Phi}^{obs} 2\Delta t \, \mathrm{diag}_j \{\delta_{jk}\} e^{2\boldsymbol{\mu} \Delta t} \\ \vdots \\ \boldsymbol{\Phi}^{obs} (\alpha - 1)\Delta t \, \mathrm{diag}_j \{\delta_{jk}\} e^{(\alpha-1)\boldsymbol{\mu} \Delta t} \end{bmatrix}, \tag{19}$$

and

$$\frac{\partial \mu_k}{\partial \theta_j} = \frac{\mu_k(\theta_j + \Delta\theta_j) - \mu_k(\theta_j)}{\Delta\theta_j} \tag{20}$$

is the eigenvalue sensibility, obtained numerically, with respect to a variation in the parameter θ_j.

4 Damage updating

When damage appears, the modal parameters change according to the damaged physical parameters. Diagnosis is performed by updating the FEM through the damaged physical parameters. It is acknowledged that natural frequencies can be measured more accurately than mode shapes. In this paper, it is assumed that the mode shape variations introduced by damage are less important than measurement errors. This allows that the eigenvectors don't need to be updated at each iteration of the Gauss-Newton algorithm. This simplification leads to significant reduction of CPU costs.

5 Case study of a fixed-free beam structure using experimental data

Through an experimental application, the updating procedure is detailed. To this end, the FEM updating method using experimental modal data is applied in two steps. In the first step, the initial FEM is tuned to the undamaged state to obtain a reference model. In the second step, the reference model is updated in order to identify the damage.

5.1 Experimental description

The experimental studied mechanical structure is a beam. Its geometrical properties are a length of $L = 34.6\ cm$, a width of $l = 2.49\ cm$, a breath of $e = 0.53\ cm$ and a density of $\rho = 7850\ kg.m^{-3}$. Furthermore, the second moment of area is calculated with the relation $I = l \times e^3/12$. A part of the beam ($4.6\ cm$) is constrained in a vise as shown in Figure (1).

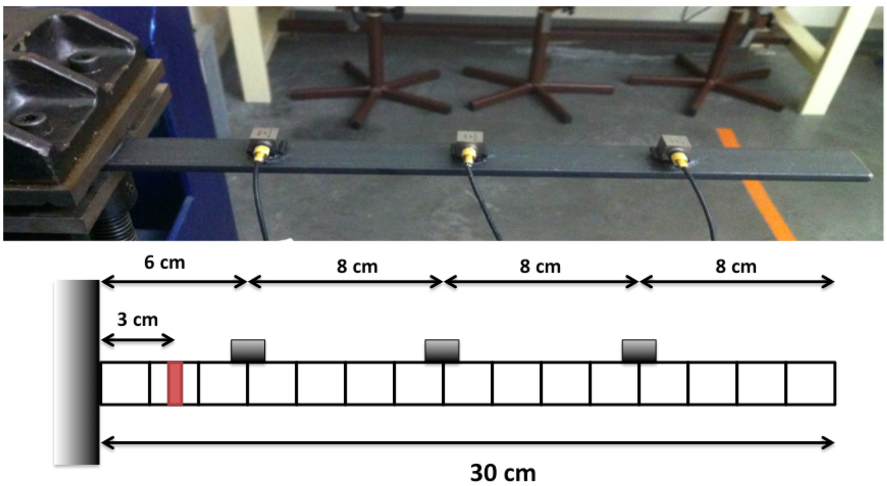

Figure 1: Structure description

The Power Spectral Density (PSD) of the vise is performed (Figure 2). It shows a natural frequency around $137\ Hz$.

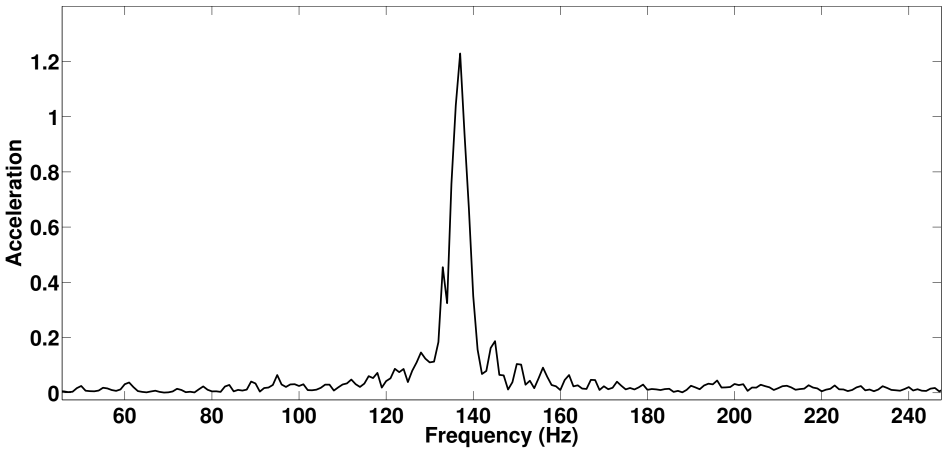

Figure 2: Frequency response of the vise

The mechanical structure is excited with an Impact Hammer. The input signal is assumed to be unknown. Output signals are recorded with three accelerometers during $10\ s$, with sampling frequency of $1.6\ kHz$. An obtained output signal is shown in Figure 2.

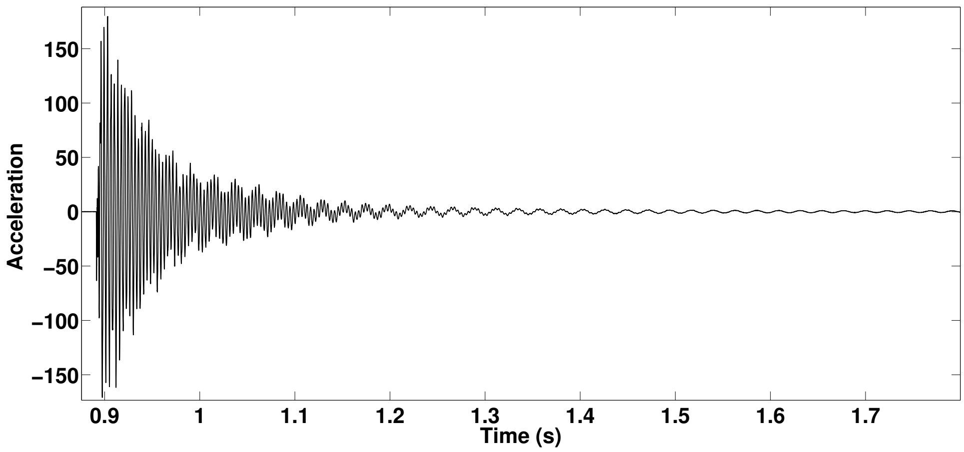

Figure 3: Output signal

This is typically an impulse response.

5.2 Stochastic subspace identification

For an overview of the dynamical behavior, a stochastic subspace identification is performed. By using the MOESP algorithm [14], the identified natural frequencies are plotted according to the model order of the experimental observability matrix, in a stabilization diagram. When a frequency is stabilized, it is assumed to have a physical meaning.

So as to better understanding, the stabilization diagram is superimposed with the PSD in Figure 4. In the frequency bandwidth $[0-800]\ Hz$ four stabilized frequencies are present, whose the frequency of the vise.

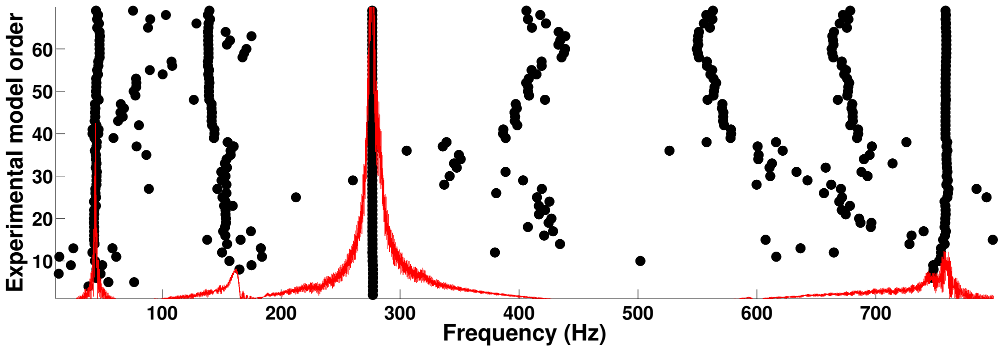

Figure 4: Stabilization diagram

5.3 Safe FEM updating

An fixed-free Euler Bernoulli cantilever beam, where only bending in a single plane, is considered. Each node has two degrees of freedom, namely the translational displacement and bending rotation. Modeling of the beam is performed by a FEM. The elementary mass and stiffness matrices are respectively obtained by

$$\mathbf{M^e} = \frac{\rho \times l \times e \times l_e}{420} \begin{bmatrix} 156 & 22l_e^2 & 54 & -13l_e \\ & 4l_e^2 & 13l_e & -3l_e^2 \\ & & 156 & -22l_e \\ \text{sym} & & & 4l_e^2 \end{bmatrix} \quad \mathbf{K^e} = E \times I \begin{bmatrix} 12/l_e^3 & 6/l_e^2 & -12/l_e^3 & 6/l_e^2 \\ & 4/l_e & -6/l_e^2 & 2/l_e \\ & & 12/l_e^3 & -6/l_e^2 \\ \text{sym} & & & 4/l_e \end{bmatrix}. \tag{21}$$

The structural damping is obtained by

$$\mathbf{D^e} = 0.01 \times \mathbf{K^e}. \tag{22}$$

The beam is modeled with 15 elements, i.e. an element length of $l_e = 2\ cm$. The previous stochastic subspace identification allowed to see that only three modes are present in the frequency band of interest. To calculate the theoretical observability matrix, only the five first modes are kept. It improves the computation time without introducing large errors. On the other hand, due to the geometrical properties of the beam, superior modes leads to an ill-conditioning of FE matrices. Among the beam proprieties, only the Young's modulus is unknown. It is updated using the method described in this article. The updating procedure is initialized with a Young's modulus value of 200 GPa, which is a typical value for this kind of material. At the end, the updated value is 167 GPa. In parallel, for better visualization of the updating results the residue used in this minimization procedure is plotted, in Figure 5, for a Young's modulus between 100 and 300 GPa. The only minimum is that reached by the optimization procedure. No local minimum between the initial and readjusted value is came distort the results.

A comparison is made between experimental and modeled natural frequencies before and after updating, for the first three modes, in Table 1.

Mode Number	Experimental	FEM			
		Before updating		After updating	
	Frequency (Hz)	Frequency (Hz)	error (%)	Frequency (Hz)	error (%)
1	43.21	48.02	11.13	43.87	1.53
2	275.6	300.9	9.18	275.0	2.2
3	756.7	842.6	11.48	769.9	1.74

Table 1: Natural frequencies comparison

At this stage, all FEM parameters are properly adjusted.

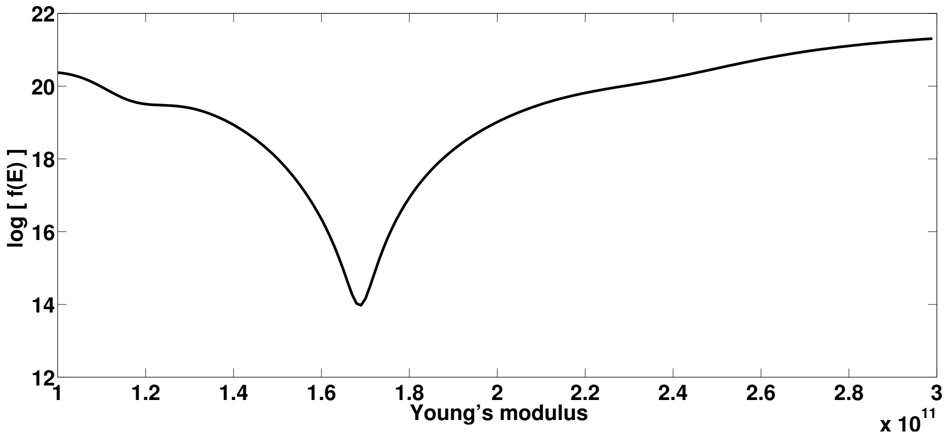

Figure 5: Young's modulus updating

5.4 Damage diagnosis

As shown in Figure 6, a damage is performed by drilling the beam to 3 cm of the embedding with a drill bit of 4 mm in diameter.

Figure 6: Performed damage

The damage is assumed to causes a loss of elementary rigidity that is sought to readjust. The different values of the residue, obtained for a variation of the element localization and percentage of stiffness reduction, are shown in Figure 7.

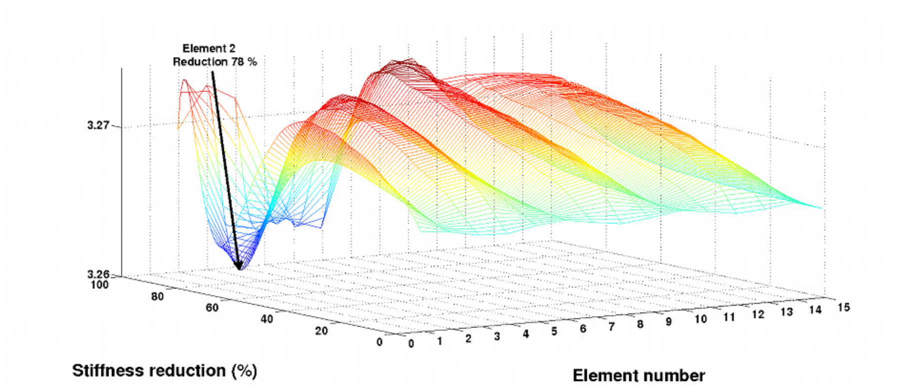

Figure 7: Minimum of the residue.

The minimum is obtained for the second element with a loss of local rigidity of $78\,\%$. The obtained localization corresponds to the created damage.

6 Conclusion

In this paper, by modifying subspace fitting method an optimized algorithm was proposed which enables the FEM of the system to be updated. Experimental experiment was presented which shown the capabilities of the method to localize faults in mechanical beam. The procedure can be extended to a structural health monitoring.

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